

UPPER TRIANGULAR MATRICES AND BILLIARD ARRAYS

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ABSTRACT. Fix a nonnegative integer d , a field \mathbb{F} , and a vector space V over \mathbb{F} with dimension $d+1$. Let T denote an invertible upper triangular matrix in $\text{Mat}_{d+1}(\mathbb{F})$. Using T we construct three flags on V . We find a necessary and sufficient condition on T for these three flags to be totally opposite. In this case, we use these three totally opposite flags to construct a Billiard Array B on V . It is known that B is determined up to isomorphism by a certain triangular array of scalar parameters called the B -values. We compute these B -values in terms of the entries of T . We describe the set of isomorphism classes of Billiard Arrays in terms of upper triangular matrices.

Keywords. Upper triangular matrix, Billiard Array, flag, Quantum group, Equitable presentation.

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1. INTRODUCTION

This paper is about a connection between upper triangular matrices and Billiard Arrays. The Billiard Array concept was introduced in [15]. This concept is closely related to the equitable presentation of $U_q(\mathfrak{sl}_2)$ [10, 15]. For more information about the equitable presentation, see [1, 3, 5–9, 12–14, 16].

We now summarize our results. Fix a nonnegative integer d , a field \mathbb{F} , and a vector space V over \mathbb{F} with dimension $d+1$. Let T denote an invertible upper triangular matrix in $\text{Mat}_{d+1}(\mathbb{F})$. View T as the transition matrix from a basis $\{u_i\}_{i=0}^d$ of V to a basis $\{v_i\}_{i=0}^d$ of V . Using T we construct three flags $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$ on V as follows. For $0 \leq i \leq d$,

$$U_i = \mathbb{F}u_0 + \mathbb{F}u_1 + \cdots + \mathbb{F}u_i = \mathbb{F}v_0 + \mathbb{F}v_1 + \cdots + \mathbb{F}v_i;$$

$$U'_i = \mathbb{F}u_d + \mathbb{F}u_{d-1} + \cdots + \mathbb{F}u_{d-i};$$

$$U''_i = \mathbb{F}v_d + \mathbb{F}v_{d-1} + \cdots + \mathbb{F}v_{d-i}.$$

In our first main result, we find a necessary and sufficient condition (called very good) on T for $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$ to be totally opposite in the sense of [15, Definition 12.1].

In [15, Theorem 12.7] it is shown how three totally opposite flags on V correspond to a Billiard Array on V . Assume that the three flags $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$ are totally opposite, and let B denote the corresponding Billiard Array on V . By [15, Lemma 19.1] B is determined up to isomorphism by a certain triangular array of scalar parameters called the B -values. In our second main result, we compute these B -values in terms of the entries of T .

Let $\mathcal{T}_d(\mathbb{F})$ denote the set of very good upper triangular matrices in $\text{Mat}_{d+1}(\mathbb{F})$. Define an equivalence relation \sim on $\mathcal{T}_d(\mathbb{F})$ as follows. For $T, T' \in \mathcal{T}_d(\mathbb{F})$, we declare $T \sim T'$ whenever there exist invertible diagonal matrices $H, K \in \text{Mat}_{d+1}(\mathbb{F})$ such

that $T' = HTK$. In our third main result, we display a bijection between the following two sets:

- (i) the equivalence classes for \sim on $\mathcal{T}_d(\mathbb{F})$;
- (ii) the isomorphism classes of Billiard Arrays on V .

We give a commutative diagram that illustrates our second and third main result. At the end of this paper, we give an example. In this example, we display a very good upper triangular matrix with entries given by q -binomial coefficients. We show that for the corresponding Billiard Array B , all the B -values are equal to q^{-1} .

The paper is organized as follows. Section 2 contains some preliminaries. Section 3 contains necessary facts about decompositions and flags. Section 4 is devoted to the correspondence between very good upper triangular matrices and totally opposite flags. This section contains our first main result. Section 5 contains necessary facts about Billiard Arrays. In Sections 6–8 we obtain our second and third main results. In Section 9, we display an example to illustrate our theory.

2. PRELIMINARIES

Throughout the paper, we fix the following notation. Let \mathbb{R} denote the field of real numbers. Recall the ring of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. Fix $d \in \mathbb{N}$. Let $\{x_i\}_{i=0}^d$ denote a sequence. We call x_i the i -component of the sequence. By the *inversion* of the sequence $\{x_i\}_{i=0}^d$ we mean the sequence $\{x_{d-i}\}_{i=0}^d$. Let \mathbb{F} denote a field. Let V denote a vector space over \mathbb{F} with dimension $d+1$. Let $\text{Mat}_{d+1}(\mathbb{F})$ denote the \mathbb{F} -algebra consisting of the $d+1$ by $d+1$ matrices that have all entries in \mathbb{F} . We index the rows and columns by $0, 1, \dots, d$. Let I denote the identity matrix in $\text{Mat}_{d+1}(\mathbb{F})$.

3. DECOMPOSITIONS AND FLAGS

In this section, we review some basic facts about decompositions and flags.

Definition 3.1. By a *decomposition* of V we mean a sequence $\{V_i\}_{i=0}^d$ consisting of one-dimensional subspaces of V such that $V = \sum_{i=0}^d V_i$ (direct sum).

Remark 3.2. For a decomposition of V , its inversion is a decomposition of V .

Example 3.3. Choose a basis $\{f_i\}_{i=0}^d$ of V . For $0 \leq i \leq d$, define $V_i = \mathbb{F}f_i$. Then $\{V_i\}_{i=0}^d$ is a decomposition of V .

Definition 3.4. Referring to Example 3.3, we say that the decomposition $\{V_i\}_{i=0}^d$ is *induced* by the basis $\{f_i\}_{i=0}^d$.

Definition 3.5. By a *flag* on V , we mean a sequence $\{W_i\}_{i=0}^d$ of subspaces of V such that W_i has dimension $i+1$ for $0 \leq i \leq d$ and $W_{i-1} \subseteq W_i$ for $1 \leq i \leq d$.

Example 3.6. Let $\{V_i\}_{i=0}^d$ denote a decomposition of V . For $0 \leq i \leq d$, define $W_i = V_0 + V_1 + \dots + V_i$. Then $\{W_i\}_{i=0}^d$ is a flag on V .

Definition 3.7. Referring to Example 3.6, we say that the flag $\{W_i\}_{i=0}^d$ is *induced* by the decomposition $\{V_i\}_{i=0}^d$.

Definition 3.8. Consider a basis of V . That basis induces a decomposition of V , which in turn induces a flag on V . We say that flag is *induced* by the given basis.

Lemma 3.9. [15, Section 6] Suppose that we are given two flags on V , denoted by $\{W_i\}_{i=0}^d$ and $\{W'_i\}_{i=0}^d$. Then the following are equivalent:

- (i) $W_i \cap W'_j = 0$ for $i + j < d$ ($0 \leq i, j \leq d$);
- (ii) there exists a decomposition $\{V_i\}_{i=0}^d$ of V that induces $\{W_i\}_{i=0}^d$ and whose inversion induces $\{W'_i\}_{i=0}^d$.

Moreover, suppose (i), (ii) hold. Then $V_i = W_i \cap W'_{d-i}$ for $0 \leq i \leq d$.

Definition 3.10. Referring to Lemma 3.9, the flags $\{W_i\}_{i=0}^d$ and $\{W'_i\}_{i=0}^d$ are called *opposite* whenever (i), (ii) hold.

We mention a variation on Lemma 3.9.

Lemma 3.11. [15, Section 6] Suppose that we are given two flags on V , denoted by $\{W_i\}_{i=0}^d$ and $\{W'_i\}_{i=0}^d$. Then they are opposite if and only if $W_i \cap W'_j = 0$ for $i + j = d - 1$ ($0 \leq i, j \leq d - 1$).

Definition 3.12. Suppose that we are given three flags on V , denoted by $\{W_i\}_{i=0}^d$, $\{W'_i\}_{i=0}^d$, $\{W''_i\}_{i=0}^d$. These flags are said to be *totally opposite* whenever $W_{d-r} \cap W'_{d-s} \cap W''_{d-t} = 0$ for all r, s, t ($0 \leq r, s, t \leq d$) such that $r + s + t > d$.

Lemma 3.13. [15, Theorem 12.3] Suppose that we are given three flags on V , denoted by $\{W_i\}_{i=0}^d$, $\{W'_i\}_{i=0}^d$, $\{W''_i\}_{i=0}^d$. Then the following are equivalent:

- (i) The flags $\{W_i\}_{i=0}^d$, $\{W'_i\}_{i=0}^d$, $\{W''_i\}_{i=0}^d$ are totally opposite.
- (ii) For $0 \leq n \leq d$, the sequences $\{W_i\}_{i=0}^{d-n}$, $\{W_{d-n} \cap W'_{n+i}\}_{i=0}^{d-n}$, $\{W_{d-n} \cap W''_{n+i}\}_{i=0}^{d-n}$ are mutually opposite flags on W_{d-n} .
- (iii) For $0 \leq n \leq d$, the sequences $\{W'_i\}_{i=0}^{d-n}$, $\{W'_{d-n} \cap W''_{n+i}\}_{i=0}^{d-n}$, $\{W'_{d-n} \cap W_{n+i}\}_{i=0}^{d-n}$ are mutually opposite flags on W'_{d-n} .
- (iv) For $0 \leq n \leq d$, the sequences $\{W''_i\}_{i=0}^{d-n}$, $\{W''_{d-n} \cap W_{n+i}\}_{i=0}^{d-n}$, $\{W''_{d-n} \cap W'_{n+i}\}_{i=0}^{d-n}$ are mutually opposite flags on W''_{d-n} .

For more information about flags, we refer the reader to [11] and [4].

4. UPPER TRIANGULAR MATRICES AND FLAGS

In this section, we explore the relation between upper triangular matrices and flags. First, we introduce some notation.

Definition 4.1. For a matrix $A \in \text{Mat}_{d+1}(\mathbb{F})$, we define some submatrices of A as follows. For $0 \leq i \leq j \leq d$, let $A[i, j]$ denote the submatrix $\{A_{kl}\}_{0 \leq k \leq j-i, i \leq l \leq j}$ of A . Note that $A[0, d] = A$.

Definition 4.2. For a matrix $A \in \text{Mat}_{d+1}(\mathbb{F})$ and $0 \leq j \leq d$, we call the submatrix $A[0, j]$ the *j-th leading principal submatrix* of A .

Definition 4.3. For a matrix $A \in \text{Mat}_{d+1}(\mathbb{F})$, we call it *good* whenever the submatrix $A[i, d]$ is invertible for $0 \leq i \leq d$.

Definition 4.4. For a matrix $A \in \text{Mat}_{d+1}(\mathbb{F})$, we call it *very good* whenever the submatrix $A[i, j]$ is invertible for $0 \leq i \leq j \leq d$.

Lemma 4.5. A matrix in $\text{Mat}_{d+1}(\mathbb{F})$ is very good if and only if each of its leading principal submatrices is good.

Proof. By Definitions 4.1–4.4. □

Referring to Definition 4.1, we now consider the case in which A is upper triangular.

Lemma 4.6. *For an upper triangular matrix $A \in \text{Mat}_{d+1}(\mathbb{F})$, the submatrix $A[0, j]$ is upper triangular for $0 \leq j \leq d$.*

Proof. By Definition 4.1. \square

Lemma 4.7. *For an invertible upper triangular matrix $A \in \text{Mat}_{d+1}(\mathbb{F})$, the submatrix $A[0, j]$ is upper triangular and invertible for $0 \leq j \leq d$.*

Proof. By Definition 4.1. \square

Consider an invertible upper triangular matrix $T \in \text{Mat}_{d+1}(\mathbb{F})$. View T as the transition matrix from a basis $\{u_i\}_{i=0}^d$ of V to a basis $\{v_i\}_{i=0}^d$ of V . Thus for $0 \leq j \leq d$,

$$(4.1) \quad v_j = \sum_{i=0}^d T_{ij} u_i.$$

For the moment, pick $x \in V$. Then there exist scalars $\{b_i(x)\}_{i=0}^d$ in \mathbb{F} such that

$$(4.2) \quad x = \sum_{i=0}^d b_i(x) u_i.$$

Moreover, there exist scalars $\{c_i(x)\}_{i=0}^d$ in \mathbb{F} such that

$$(4.3) \quad x = \sum_{i=0}^d c_i(x) v_i.$$

By (4.1)–(4.3),

$$(4.4) \quad Tc = b,$$

where $c = (c_0(x), c_1(x), \dots, c_d(x))^t$ and $b = (b_0(x), b_1(x), \dots, b_d(x))^t$.

We now use T to construct three flags on V .

Lemma 4.8. *With the above notation, the following two flags on V coincide:*

- (i) *the flag induced by $\{u_i\}_{i=0}^d$;*
- (ii) *the flag induced by $\{v_i\}_{i=0}^d$.*

Proof. By (4.1) and since T is upper triangular. \square

We now define three flags on V , denoted by $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$. The flag $\{U_i\}_{i=0}^d$ is induced by the basis $\{u_i\}_{i=0}^d$ or $\{v_i\}_{i=0}^d$. The flag $\{U'_i\}_{i=0}^d$ (resp. $\{U''_i\}_{i=0}^d$) is induced by the basis $\{u_{d-i}\}_{i=0}^d$ (resp. $\{v_{d-i}\}_{i=0}^d$). More explicitly, for $0 \leq i \leq d$,

$$(4.5) \quad U_i = \mathbb{F}u_0 + \mathbb{F}u_1 + \dots + \mathbb{F}u_i = \mathbb{F}v_0 + \mathbb{F}v_1 + \dots + \mathbb{F}v_i;$$

$$(4.6) \quad U'_i = \mathbb{F}u_d + \mathbb{F}u_{d-1} + \dots + \mathbb{F}u_{d-i};$$

$$(4.7) \quad U''_i = \mathbb{F}v_d + \mathbb{F}v_{d-1} + \dots + \mathbb{F}v_{d-i}.$$

By Lemma 3.9, the flag $\{U_i\}_{i=0}^d$ is opposite to the flags $\{U'_i\}_{i=0}^d$ and $\{U''_i\}_{i=0}^d$. Our next goal is to give a necessary and sufficient condition for the flags $\{U'_i\}_{i=0}^d$ and $\{U''_i\}_{i=0}^d$ to be opposite. We will use the following lemma.

Lemma 4.9. *With the above notation, for $0 \leq i \leq d-1$, $U'_i \cap U''_{d-1-i} = 0$ if and only if $\det(T[i+1, d]) \neq 0$.*

Proof. Consider $x \in V$. We refer to the notation around (4.2) and (4.3). We make two observations about x . The first observation is that by (4.6), we have $x \in U'_i$ if and only if $b_n(x) = 0$ for $0 \leq n \leq d-1-i$. The second observation is that by (4.7), we have $x \in U''_{d-1-i}$ if and only if $c_n(x) = 0$ for $0 \leq n \leq i$. In this case, by (4.4),

$$(4.8) \quad T[i+1, d](c_{i+1}(x), c_{i+2}(x), \dots, c_d(x))^t = (b_0(x), b_1(x), \dots, b_{d-1-i}(x))^t.$$

First assume that $\det(T[i+1, d]) \neq 0$. We will show that $U'_i \cap U''_{d-1-i} = 0$. To do this, we assume $x \in U'_i \cap U''_{d-1-i}$, and show that $x = 0$. By the first observation, $b_n(x) = 0$ for $0 \leq n \leq d-1-i$. By the second observation, $c_n(x) = 0$ for $0 \leq n \leq i$ and (4.8) holds. By these comments,

$$(4.9) \quad T[i+1, d](c_{i+1}(x), c_{i+2}(x), \dots, c_d(x))^t = 0.$$

By (4.9) and $\det(T[i+1, d]) \neq 0$, the vector $(c_{i+1}(x), c_{i+2}(x), \dots, c_d(x))^t = 0$. In other words, $c_n(x) = 0$ for $i+1 \leq n \leq d$. We have shown that $c_n(x) = 0$ for $0 \leq n \leq d$. Hence $x = 0$. Therefore $U'_i \cap U''_{d-1-i} = 0$.

Next assume that $\det(T[i+1, d]) = 0$. We will show that $U'_i \cap U''_{d-1-i} \neq 0$. By the assumption and linear algebra, there exists a nonzero vector

$$w = (w_{i+1}, w_{i+2}, \dots, w_d) \in \mathbb{F}^{d-i}$$

such that $T[i+1, d]w^t = 0$. Choose the vector x such that $c_n(x) = 0$ for $0 \leq n \leq i$ and $c_n(x) = w_n$ for $i+1 \leq n \leq d$. Observe that $x \neq 0$ and satisfies (4.9). By the second observation, $x \in U''_{d-1-i}$ and (4.8) holds. By (4.8) and (4.9), $b_n(x) = 0$ for $0 \leq n \leq d-1-i$. By the first observation, $x \in U'_i$. We have shown that $0 \neq x \in U'_i \cap U''_{d-1-i}$. Therefore $U'_i \cap U''_{d-1-i} \neq 0$. \square

Proposition 4.10. *The flags $\{U'_i\}_{i=0}^d$ and $\{U''_i\}_{i=0}^d$ are opposite if and only if T is good in the sense of Definition 4.3.*

Proof. Recall from Definition 4.1 that $T[0, d] = T$. Since T is invertible, we obtain $\det(T[0, d]) \neq 0$. Therefore, by Definition 4.3, T is good if and only if $\det(T[i+1, d]) \neq 0$ for $0 \leq i \leq d-1$. By Lemma 4.9, this happens if and only if $U'_i \cap U''_{d-1-i} = 0$ for $0 \leq i \leq d-1$. The result follows in view of Lemma 3.11. \square

Corollary 4.11. *T is good if and only if T^{-1} is good.*

Proof. Going through the construction around (4.5)–(4.7) using T , we obtain a sequence of three flags $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$. Repeating the construction with T replaced by T^{-1} , we obtain a sequence of three flags $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$. The result follows in view of Proposition 4.10. \square

Next, we give a necessary and sufficient condition for the three flags $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$ to be totally opposite in the sense of Definition 3.12.

Proposition 4.12. *The three flags $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$ are totally opposite in the sense of Definition 3.12 if and only if T is very good in the sense of Definition 4.4.*

Proof. By parts (i), (ii) of Lemma 3.13 along with Definition 4.4 and Lemma 4.5, it suffices to show that for $0 \leq n \leq d$, the sequences $\{U_i\}_{i=0}^{d-n}$, $\{U_{d-n} \cap U'_{n+i}\}_{i=0}^{d-n}$, $\{U_{d-n} \cap U''_{n+i}\}_{i=0}^{d-n}$ are mutually opposite flags on U_{d-n} if and only if $\det(T[i, d -$

$n]) \neq 0$ for $0 \leq i \leq d - n$. Let n be given. By (4.5)–(4.7) we find that for $0 \leq i \leq d - n$,

$$(4.10) \quad U_{d-n} \cap U'_{n+i} = \mathbb{F}u_{d-n-i} + \mathbb{F}u_{d-n-i+1} + \cdots + \mathbb{F}u_{d-n};$$

$$(4.11) \quad U_{d-n} \cap U''_{n+i} = \mathbb{F}v_{d-n-i} + \mathbb{F}v_{d-n-i+1} + \cdots + \mathbb{F}v_{d-n}.$$

By (4.5), (4.10) and Lemma 3.9, the flag $\{U_i\}_{i=0}^{d-n}$ on U_{d-n} is opposite to the flag $\{U_{d-n} \cap U'_{n+i}\}_{i=0}^{d-n}$ on U_{d-n} . Similarly, by (4.5), (4.11) and Lemma 3.9, the flag $\{U_i\}_{i=0}^{d-n}$ on U_{d-n} is opposite to the flag $\{U_{d-n} \cap U''_{n+i}\}_{i=0}^{d-n}$ on U_{d-n} . By Lemma 4.7, the submatrix $T[0, d - n]$ is invertible and upper triangular. Therefore we can apply Proposition 4.10 to the two flags $\{U_{d-n} \cap U'_{n+i}\}_{i=0}^{d-n}$ and $\{U_{d-n} \cap U''_{n+i}\}_{i=0}^{d-n}$ on U_{d-n} . By this, the two flags $\{U_{d-n} \cap U'_{n+i}\}_{i=0}^{d-n}$ and $\{U_{d-n} \cap U''_{n+i}\}_{i=0}^{d-n}$ are opposite if and only if $\det(T[i, d - n]) \neq 0$ for $0 \leq i \leq d - n$. We have shown that the sequences $\{U_i\}_{i=0}^{d-n}$, $\{U_{d-n} \cap U'_{n+i}\}_{i=0}^{d-n}$, $\{U_{d-n} \cap U''_{n+i}\}_{i=0}^{d-n}$ are mutually opposite flags on U_{d-n} if and only if $\det(T[i, d - n]) \neq 0$ for $0 \leq i \leq d - n$. The result follows. \square

Corollary 4.13. *T is very good if and only if T^{-1} is very good.*

Proof. Similar to the proof of Corollary 4.11. \square

Definition 4.14. Let $\mathcal{T}_d(\mathbb{F})$ denote the set of very good upper triangular matrices in $\text{Mat}_{d+1}(\mathbb{F})$. Note that each element of $\mathcal{T}_d(\mathbb{F})$ is invertible.

5. BILLIARD ARRAYS

In this section, we develop some results about Billiard Arrays that will be used later in the paper. We will refer to the following basis for the vector space \mathbb{R}^3 :

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1).$$

Define a subset $\Phi = \{e_i - e_j \mid 1 \leq i, j \leq 3, i \neq j\}$ of \mathbb{R}^3 . The set Φ is often called the root system A_2 . For notational convenience define

$$(5.1) \quad \alpha = e_1 - e_2, \quad \beta = e_2 - e_3, \quad \gamma = e_3 - e_1.$$

Note that

$$\Phi = \{\pm\alpha, \pm\beta, \pm\gamma\}, \quad \alpha + \beta + \gamma = 0.$$

Definition 5.1. Let Δ_d denote the subset of \mathbb{R}^3 consisting of the three-tuples of natural numbers whose sum is d . Thus

$$\Delta_d = \{(r, s, t) \mid r, s, t \in \mathbb{N}, r + s + t = d\}.$$

Remark 5.2. We can arrange the elements of Δ_d in a triangular array. For example, if $d = 3$, the array looks as follows after deleting all punctuation:

$$\begin{array}{cccc} & & 030 & \\ & 120 & 021 & \\ 210 & 111 & 012 & \\ 300 & 201 & 102 & 003 \end{array}$$

An element in Δ_d is called a *location*.

Definition 5.3. For $\eta \in \{1, 2, 3\}$, the η -corner of Δ_d is the location in Δ_d that has η -coordinate d and all other coordinates 0. By a *corner* of Δ_d we mean the 1-corner or 2-corner or 3-corner. The corners in Δ_d are listed below.

$$de_1 = (d, 0, 0), \quad de_2 = (0, d, 0), \quad de_3 = (0, 0, d).$$

Definition 5.4. For $\eta \in \{1, 2, 3\}$, the η -boundary of Δ_d is the set of locations in Δ_d that have η -coordinate 0. The *boundary* of Δ_d is the union of its 1-boundary, 2-boundary and 3-boundary. By the *interior* of Δ_d we mean the set of locations in Δ_d that are not on the boundary.

Example 5.5. Referring to the picture of Δ_3 from Remark 5.2, the 2-boundary of Δ_3 consists of the four locations in the bottom row.

Definition 5.6. For $\eta \in \{1, 2, 3\}$ we define a binary relation on Δ_d called η -collinearity. By definition, locations λ, λ' in Δ_d are η -collinear whenever the η -coordinate of $\lambda - \lambda'$ is 0. Note that η -collinearity is an equivalence relation. Each equivalence class will be called an η -line. By a *line* in Δ_d we mean a 1-line or 2-line or 3-line.

Example 5.7. Referring to the picture of Δ_3 from Remark 5.2, the horizontal rows are the 2-lines of Δ_3 .

Definition 5.8. Locations λ, μ in Δ_d are called *adjacent* whenever $\lambda - \mu \in \Phi$.

Definition 5.9. By an *edge* in Δ_d we mean a set of two adjacent locations.

Definition 5.10. By a *3-clique* in Δ_d we mean a set of three mutually adjacent locations in Δ_d . There are two kinds of 3-cliques: $\triangle(\text{black})$ and $\nabla(\text{white})$.

Lemma 5.11. [15, Lemma 4.31] Assume $d \geq 1$. We describe a bijection from Δ_{d-1} to the set of black 3-cliques in Δ_d . The bijection sends each $(r, s, t) \in \Delta_{d-1}$ to the black 3-clique in Δ_d consisting of the locations $(r+1, s, t), (r, s+1, t), (r, s, t+1)$.

Lemma 5.12. [15, Lemma 4.32] Assume $d \geq 2$. We describe a bijection from Δ_{d-2} to the set of white 3-cliques in Δ_d . The bijection sends each $(r, s, t) \in \Delta_{d-2}$ to the white 3-clique in Δ_d consisting of the locations $(r, s+1, t+1), (r+1, s, t+1), (r+1, s+1, t)$.

Lemma 5.13. [15, Lemma 4.33] For Δ_d , each edge is contained in a unique black 3-clique and at most one white 3-clique.

Let $\mathcal{P}_1(V)$ denote the set of 1-dimensional subspaces of V .

Definition 5.14. [15, Definition 7.1] By a *Billiard Array* on V we mean a function $B : \Delta_d \rightarrow \mathcal{P}_1(V)$, $\lambda \mapsto B_\lambda$ that satisfies the following conditions:

- (i) for each line L in Δ_d the sum $\sum_{\lambda \in L} B_\lambda$ is direct;
- (ii) for each black 3-clique C in Δ_d the sum $\sum_{\lambda \in C} B_\lambda$ is not direct.

We say that B is *over* \mathbb{F} . We call V the *underlying vector space*. We call d the *diameter* of B .

Lemma 5.15. [15, Corollary 7.4] Let B denote a Billiard Array on V . Let λ, μ, ν denote the locations in Δ_d that form a black 3-clique. Then each of B_λ, B_μ, B_ν is contained in the sum of the other two.

Definition 5.16. Let V' denote a vector space over \mathbb{F} with dimension $d+1$. Let B (resp. B') denote a Billiard Array on V (resp. V'). By an *isomorphism* of Billiard Arrays from B to B' we mean an \mathbb{F} -linear bijection $\sigma : V \rightarrow V'$ that sends $B_\lambda \mapsto B'_\lambda$ for all $\lambda \in \Delta_d$. The Billiard Arrays B and B' are called *isomorphic* whenever there exists an isomorphism of Billiard Arrays from B to B' .

From now until the end of Lemma 5.21, let B denote a Billiard Array on V .

Definition 5.17. Pick $\eta \in \{1, 2, 3\}$. Following [15, Section 9] we now define a flag on V called the B -flag $[\eta]$. For $0 \leq i \leq d$, the i -component of this flag is $\sum_\lambda B_\lambda$, where the sum is over all $\lambda \in \Delta_d$ that have η -coordinate at least $d-i$.

Definition 5.18. Pick distinct $\eta, \xi \in \{1, 2, 3\}$. Following [15, Section 10] we now define a decomposition of V called the B -decomposition $[\eta, \xi]$. For $0 \leq i \leq d$ the i -component of this decomposition is the subspace B_λ , where the location λ is described in the table below:

η	ξ	λ_i
1	2	$(d-i, i, 0)$
2	1	$(i, d-i, 0)$
2	3	$(0, d-i, i)$
3	2	$(0, i, d-i)$
3	1	$(i, 0, d-i)$
1	3	$(d-i, 0, i)$

Lemma 5.19. [15, Lemma 10.6] For distinct $\eta, \xi \in \{1, 2, 3\}$ the B -decomposition $[\eta, \xi]$ of V induces the B -flag $[\eta]$ on V .

Lemma 5.20. [15, Theorem 12.4] The B -flags $[1]$, $[2]$, $[3]$ on V from Definition 5.17 are totally opposite in the sense of Definition 3.12.

Lemma 5.21. [15, Corollary 11.2] Pick a location $\lambda = (r, s, t)$ in Δ_d . Then B_λ is equal to the intersection of the following three sets:

- (i) component $d-r$ of the B -flag $[1]$;
- (ii) component $d-s$ of the B -flag $[2]$;
- (iii) component $d-t$ of the B -flag $[3]$.

Theorem 5.22. [15, Theorem 12.7] Suppose that we are given three totally opposite flags on V , denoted by $\{W_i\}_{i=0}^d$, $\{W'_i\}_{i=0}^d$, $\{W''_i\}_{i=0}^d$. For each location $\lambda = (r, s, t)$ in Δ_d , define $B_\lambda = W_{d-r} \cap W'_{d-s} \cap W''_{d-t}$. Then the map $B : \Delta_d \rightarrow \mathcal{P}_1(V)$, $\lambda \mapsto B_\lambda$ is a Billiard Array on V .

Definition 5.23. [15, Definition 8.1] By a *Concrete Billiard Array* on V we mean a function $\mathcal{B} : \Delta_d \rightarrow V$, $\lambda \mapsto \mathcal{B}_\lambda$ that satisfies the following conditions:

- (i) for each line L in Δ_d the vectors $\{\mathcal{B}_\lambda\}_{\lambda \in L}$ are linearly independent;
- (ii) for each black 3-clique C in Δ_d the vectors $\{\mathcal{B}_\lambda\}_{\lambda \in C}$ are linearly dependent.

We say that \mathcal{B} is *over* \mathbb{F} . We call V the *underlying vector space*. We call d the *diameter* of \mathcal{B} .

Example 5.24. Let B denote a Billiard Array on V . For $\lambda \in \Delta_d$ pick $0 \neq \mathcal{B}_\lambda \in B_\lambda$. Then the function $\mathcal{B} : \Delta_d \rightarrow V$, $\lambda \mapsto \mathcal{B}_\lambda$ is a Concrete Billiard Array on V .

Definition 5.25. Let B denote a Billiard Array on V , and let \mathcal{B} denote a concrete Billiard Array on V . We say that B, \mathcal{B} *correspond* whenever \mathcal{B}_λ spans B_λ for all $\lambda \in \Delta_d$.

Definition 5.26. Let λ, μ denote locations in Δ_d that form an edge. By Lemma 5.13 there exists a unique location $\nu \in \Delta_d$ such that λ, μ, ν form a black 3-clique. We call ν the *completion* of the edge.

From now until the end of Definition 5.33, let B denote a Billiard Array on V .

Definition 5.27. Let λ, μ denote locations in Δ_d that form an edge. By a *brace* for this edge, we mean a set of nonzero vectors $u \in B_\lambda, v \in B_\mu$ such that $u + v \in B_\nu$. Here ν denotes the completion of the edge.

Lemma 5.28. [15, Lemma 13.12] Let λ, μ denote locations in Δ_d that form an edge. Then each nonzero $u \in B_\lambda$ is contained in a unique brace for this edge.

Definition 5.29. [15, Definition 14.1] Let λ, μ denote adjacent locations in Δ_d . We define an \mathbb{F} -linear map $\tilde{B}_{\lambda,\mu} : B_\lambda \rightarrow B_\mu$ as follows. For each brace $u \in B_\lambda, v \in B_\mu$, the map $\tilde{B}_{\lambda,\mu}$ sends $u \mapsto v$. The map $\tilde{B}_{\lambda,\mu}$ is well defined by Lemma 5.28. Observe that $\tilde{B}_{\lambda,\mu} : B_\lambda \rightarrow B_\mu$ is bijective.

Definition 5.30. [15, Definition 14.9] Let λ, μ, ν denote locations in Δ_d that form a white 3-clique. Then the composition

$$B_\lambda \xrightarrow{\tilde{B}_{\lambda,\mu}} B_\mu \xrightarrow{\tilde{B}_{\mu,\nu}} B_\nu \xrightarrow{\tilde{B}_{\nu,\lambda}} B_\lambda$$

is a nonzero scalar multiple of the identity map on B_λ . The scalar is called the *clockwise B-value* (resp. *counterclockwise B-value*) of the clique whenever λ, μ, ν runs clockwise (resp. counterclockwise) around the clique.

Definition 5.31. For each white 3-clique in Δ_d , by its *B-value* we mean its clockwise B-value.

Definition 5.32. By a *value function* on Δ_d , we mean a function $\psi : \Delta_d \rightarrow \mathbb{F} \setminus \{0\}$.

Definition 5.33. [15, Definition 14.13] Assume $d \geq 2$. We define a function $\hat{B} : \Delta_{d-2} \rightarrow \mathbb{F}$ as follows. Pick $(r, s, t) \in \Delta_{d-2}$. To describe the image of (r, s, t) under \hat{B} , consider the corresponding white 3-clique in Δ_d from Lemma 5.12. The B-value of this 3-clique is the image of (r, s, t) under \hat{B} . Observe that \hat{B} is a value function on Δ_{d-2} in the sense of Definition 5.32. We call \hat{B} the *value function* for B .

Definition 5.34. Let $BA_d(\mathbb{F})$ denote the set of isomorphism classes of Billiard Arrays over \mathbb{F} that have diameter d .

Definition 5.35. Let $VF_d(\mathbb{F})$ denote the set of value functions on Δ_d .

Definition 5.36. Assume $d \geq 2$. We now define a map $\theta : BA_d(\mathbb{F}) \rightarrow VF_{d-2}(\mathbb{F})$. For $B \in BA_d(\mathbb{F})$, the image of B under θ is the value function \hat{B} from Definition 5.33.

Lemma 5.37. [15, Lemma 19.1] Assume $d \geq 2$. Then the map $\theta : BA_d(\mathbb{F}) \rightarrow VF_{d-2}(\mathbb{F})$ from Definition 5.36 is bijective.

Definition 5.38. Let \mathcal{B} denote a Concrete Billiard Array on V . Let B denote the corresponding Billiard Array on V from Definition 5.25. Let λ, μ denote adjacent locations in Δ_d . Recall the bijection $\tilde{B}_{\lambda, \mu} : B_\lambda \rightarrow B_\mu$ from Definition 5.29. Recall that \mathcal{B}_λ is a basis for B_λ and \mathcal{B}_μ is a basis for B_μ . Define a scalar $\tilde{\mathcal{B}}_{\lambda, \mu} \in \mathbb{F}$ such that $\tilde{B}_{\lambda, \mu}$ sends $\mathcal{B}_\lambda \mapsto \tilde{\mathcal{B}}_{\lambda, \mu} \mathcal{B}_\mu$. Note that $\tilde{\mathcal{B}}_{\lambda, \mu} \neq 0$.

Lemma 5.39. [15, Lemma 15.6] Let \mathcal{B} denote a Concrete Billiard Array on V . Let λ, μ, ν denote locations in Δ_d that form a black 3-clique. Then

$$\mathcal{B}_\lambda + \tilde{\mathcal{B}}_{\lambda, \mu} \mathcal{B}_\mu + \tilde{\mathcal{B}}_{\lambda, \nu} \mathcal{B}_\nu = 0.$$

Lemma 5.40. [15, Lemma 15.9] Let \mathcal{B} denote a Concrete Billiard Array on V . Let B denote the corresponding Billiard Array on V from Definition 5.25. Let λ, μ, ν denote the locations in Δ_d that form a white 3-clique. Then the clockwise (resp. counterclockwise) B -value of the clique is equal to

$$\tilde{\mathcal{B}}_{\lambda, \mu} \tilde{\mathcal{B}}_{\mu, \nu} \tilde{\mathcal{B}}_{\nu, \lambda}$$

whenever the sequence λ, μ, ν runs clockwise (resp. counterclockwise) around the clique.

Next we consider the 2-boundary of Δ_d .

Proposition 5.41. Given a Billiard Array B on V , let $\{V_i\}_{i=0}^d$ denote the B -decomposition $[1, 3]$ of V from Definition 5.18. Then for $\lambda = (r, s, t) \in \Delta_d$,

$$(5.2) \quad B_\lambda \subseteq V_t + V_{t+1} + \cdots + V_{d-r}.$$

Proof. We do induction on s . First assume that $s = 0$. Then by Definition 5.18, $B_\lambda = V_t$. Next assume that $s > 0$. Consider the black 3-clique in Δ_d with the locations $\lambda = (r, s, t), \mu = (r, s-1, t+1), \nu = (r+1, s-1, t)$. By induction,

$$(5.3) \quad B_\mu \subseteq V_{t+1} + V_{t+2} + \cdots + V_{d-r};$$

$$(5.4) \quad B_\nu \subseteq V_t + V_{t+1} + \cdots + V_{d-r-1}.$$

By Lemma 5.15,

$$(5.5) \quad B_\lambda \subseteq B_\mu + B_\nu.$$

The equation (5.2) follows from (5.3)–(5.5). \square

Lemma 5.42. Let B denote a Billiard Array on V . Let $\{u_i\}_{i=0}^d$ (resp. $\{v_i\}_{i=0}^d$) denote a basis of V that induces the B -decomposition $[1, 2]$ (resp. B -decomposition $[1, 3]$). Then the transition matrices between $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ are upper triangular.

Proof. By Proposition 5.41. \square

We next show that the transition matrices in Lemma 5.42 are very good in the sense of Definition 4.4. By Corollary 4.13, it suffices to show that the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ is very good.

Lemma 5.43. Referring to Lemma 5.42, let T denote the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. Then T is very good in the sense of Definition 4.4.

Proof. Consider the corresponding three flags $\{U_i\}_{i=0}^d, \{U'_i\}_{i=0}^d, \{U''_i\}_{i=0}^d$ on V from (4.5)–(4.7). By Lemma 5.19, the B -flag $[1]$ (resp. $[2]$) (resp. $[3]$) is the flag $\{U_i\}_{i=0}^d$ (resp. $\{U'_i\}_{i=0}^d$) (resp. $\{U''_i\}_{i=0}^d$). By Lemma 5.20, the three flags $\{U_i\}_{i=0}^d, \{U'_i\}_{i=0}^d, \{U''_i\}_{i=0}^d$ are totally opposite. By Proposition 4.12, T is very good. \square

Recall the set $\mathcal{T}_d(\mathbb{F})$ from Definition 4.14.

Corollary 5.44. *Referring to Lemma 5.43, $T \in \mathcal{T}_d(\mathbb{F})$.*

Proof. By Lemmas 5.42, 5.43. \square

Definition 5.45. For location $\tau = (r, s, t) \in \Delta_{d-1}$, consider the corresponding black 3-clique in Δ_d from Lemma 5.11, with locations $\lambda = (r+1, s, t), \mu = (r, s+1, t), \nu = (r, s, t+1)$. Given a Concrete Billiard Array \mathcal{B} on V , we say that \mathcal{B} is τ -standard whenever $\mathcal{B}_\lambda - \mathcal{B}_\mu \in \mathbb{F}\mathcal{B}_\nu$. We call \mathcal{B} standard whenever \mathcal{B} is τ -standard for all $\tau \in \Delta_{d-1}$.

Let B denote a Billiard Array on V . For $0 \leq i \leq d$, pick $0 \neq f_i \in B_\kappa$ where $\kappa = (d-i, 0, i) \in \Delta_d$. Observe that $\{f_i\}_{i=0}^d$ is a basis of V that induces the B -decomposition $[1, 3]$.

Lemma 5.46. *With the above notation, there exists a unique standard Concrete Billiard Array \mathcal{B} on V such that*

- (i) \mathcal{B} corresponds to B in the sense of Definition 5.25;
- (ii) for $0 \leq i \leq d$, $\mathcal{B}_\kappa = f_i$ where $\kappa = (d-i, 0, i) \in \Delta_d$.

Proof. First we show that \mathcal{B} exists. Let $\lambda = (r, s, t) \in \Delta_d$. We construct \mathcal{B}_λ by induction on s . For $s = 0$ define $\mathcal{B}_\lambda = f_t$. By the construction $0 \neq \mathcal{B}_\lambda \in B_\lambda$. Next assume that $s > 0$. Consider the black 3-clique in Δ_d with the locations $\lambda = (r, s, t), \mu = (r, s-1, t+1), \nu = (r+1, s-1, t)$. The vectors \mathcal{B}_μ and \mathcal{B}_ν have been determined by the induction. By Lemma 5.28, there exists a nonzero vector $\mathcal{B}_\lambda \in B_\lambda$ such that $\mathcal{B}_\lambda - \mathcal{B}_\nu \in \mathbb{F}\mathcal{B}_\mu$. We have constructed \mathcal{B}_λ such that $0 \neq \mathcal{B}_\lambda \in B_\lambda$ for all $\lambda \in \Delta_d$. By Example 5.24 and Definition 5.25, \mathcal{B} is a Concrete Billiard Array on V that corresponds to B . By the above construction and Definition 5.45, \mathcal{B} is standard. We have shown that \mathcal{B} exists. The uniqueness of \mathcal{B} follows by (ii) and Definition 5.45. \square

Consider the standard Concrete Billiard Array \mathcal{B} from Lemma 5.46. For location $\lambda = (r, s, t) \in \Delta_d$, by Proposition 5.41 the vector \mathcal{B}_λ is a linear combination of $f_t, f_{t+1}, \dots, f_{d-r}$. Let $\{a_i(\lambda)\}_{i=t}^{d-r}$ denote the corresponding coefficients, so that

$$(5.6) \quad \mathcal{B}_\lambda = a_t(\lambda)f_t + a_{t+1}(\lambda)f_{t+1} + \dots + a_{d-r}(\lambda)f_{d-r}.$$

Lemma 5.47. *With the above notation, $a_t(\lambda) = 1$.*

Proof. We do induction on s . First assume that $s = 0$. Then by Lemma 5.46(ii), $\mathcal{B}_\lambda = f_t$. Therefore $a_t(\lambda) = 1$. Next assume that $s > 0$. Consider the black 3-clique in Δ_d with the locations $\lambda = (r, s, t), \mu = (r, s-1, t+1), \nu = (r+1, s-1, t)$. By Proposition 5.41,

$$(5.7) \quad \mathcal{B}_\nu = a_t(\nu)f_t + a_{t+1}(\nu)f_{t+1} + \dots + a_{d-r-1}(\nu)f_{d-r-1};$$

$$(5.8) \quad \mathcal{B}_\mu = a_{t+1}(\mu)f_{t+1} + a_{t+2}(\mu)f_{t+2} + \dots + a_{d-r}(\mu)f_{d-r}.$$

By induction,

$$(5.9) \quad a_t(\nu) = 1.$$

Since \mathcal{B} is standard,

$$(5.10) \quad \mathcal{B}_\lambda - \mathcal{B}_\nu \in \mathbb{F}\mathcal{B}_\mu.$$

By (5.6)–(5.10) we have $a_t(\lambda) = 1$. \square

For more information about Billiard Arrays, we refer the reader to [15].

6. UPPER TRIANGULAR MATRICES AND BILLIARD ARRAYS

Recall the set $\mathcal{T}_d(\mathbb{F})$ from Definition 4.14. In this section, we consider a matrix $T \in \mathcal{T}_d(\mathbb{F})$. Using T we construct a Billiard Array B . Then for each white 3-clique in Δ_d , we compute its B -value in terms of the entries of T .

Definition 6.1. Recall the set $BA_d(\mathbb{F})$ from Definition 5.34. We define a map $b : \mathcal{T}_d(\mathbb{F}) \rightarrow BA_d(\mathbb{F})$ as follows. Let $T \in \mathcal{T}_d(\mathbb{F})$. View T as the transition matrix from a basis $\{u_i\}_{i=0}^d$ of V to a basis $\{v_i\}_{i=0}^d$ of V as around (4.1). Consider the corresponding three flags $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$ on V from (4.5)–(4.7). These flags are totally opposite by Proposition 4.12, so they correspond to a Billiard Array on V by Theorem 5.22. Since the bases $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ are not uniquely determined, this Billiard Array is only defined up to isomorphism of Billiard Arrays. The isomorphism class of this Billiard Array is the image of T under b .

In this section, we fix $T \in \mathcal{T}_d(\mathbb{F})$. By Definition 4.14, T is upper triangular and invertible. Fix the two bases $\{u_i\}_{i=0}^d$, $\{v_i\}_{i=0}^d$ of V as around (4.1) and the three flags $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$ on V from (4.5)–(4.7). Let B denote the corresponding Billiard Array on V from Theorem 5.22. Observe that $B \in b(T)$.

Lemma 6.2. *With the above notation, B is the unique Billiard Array on V such that for $0 \leq i \leq d$,*

- (i) $B_\mu = \mathbb{F}u_i$ where $\mu = (d-i, i, 0) \in \Delta_d$;
- (ii) $B_\nu = \mathbb{F}v_i$ where $\nu = (d-i, 0, i) \in \Delta_d$.

Proof. First we show that B satisfies (i). By Theorem 5.22, $B_\mu = U_i \cap U'_{d-i} \cap U''_d$. The subspace U_i is given by (4.5). By (4.6), $U'_{d-i} = \mathbb{F}u_d + \mathbb{F}u_{d-1} + \cdots + \mathbb{F}u_i$. By (4.7), $U''_d = \mathbb{F}v_d + \mathbb{F}v_{d-1} + \cdots + \mathbb{F}v_0 = V$. By the above comments, $U_i \cap U'_{d-i} \cap U''_d = \mathbb{F}u_i$. Hence $B_\mu = \mathbb{F}u_i$. Therefore B satisfies (i). Similarly, B satisfies (ii). Next we show that B is the unique Billiard Array on V that satisfies (i) and (ii). Suppose that B' is a Billiard Array on V that satisfies (i) and (ii). By Lemma 5.19, the B' -flag [1] (resp. [2]) (resp. [3]) is the flag $\{U_i\}_{i=0}^d$ (resp. $\{U'_i\}_{i=0}^d$) (resp. $\{U''_i\}_{i=0}^d$). By Lemma 5.21 and Definition 6.1, $B_\lambda = B'_\lambda$ for all $\lambda \in \Delta_d$. Therefore $B = B'$. We conclude that B is the unique Billiard Array on V that satisfies (i) and (ii). \square

Lemma 6.3. *Consider the Billiard Array B on V . There exists a unique standard Concrete Billiard Array \mathcal{B} on V such that*

- (i) \mathcal{B} corresponds to B in the sense of Definition 5.25;
- (ii) for $0 \leq i \leq d$, $\mathcal{B}_\nu = v_i$ where $\nu = (d-i, 0, i) \in \Delta_d$.

Proof. By Lemma 5.46 and Lemma 6.2(ii). \square

Consider the Concrete Billiard Array \mathcal{B} on V from Lemma 6.3. For location $\lambda = (r, s, t)$ in Δ_d , by Proposition 5.41 the vector \mathcal{B}_λ is a linear combination of the vectors $u_s, u_{s+1}, \dots, u_{d-r}$ and also a linear combination of the vectors $v_t, v_{t+1}, \dots, v_{d-r}$. For notational convenience, abbreviate $b_i(\lambda) = b_i(\mathcal{B}_\lambda)$ in (4.2) and $c_i(\lambda) = c_i(\mathcal{B}_\lambda)$ in (4.3), so that

$$(6.1) \quad \mathcal{B}_\lambda = b_s(\lambda)u_s + b_{s+1}(\lambda)u_{s+1} + \cdots + b_{d-r}(\lambda)u_{d-r};$$

$$(6.2) \quad \mathcal{B}_\lambda = c_t(\lambda)v_t + c_{t+1}(\lambda)v_{t+1} + \cdots + c_{d-r}(\lambda)v_{d-r}.$$

In the following result we compute the coefficients in (6.2) in terms of the entries of T . The coefficients in (6.1) can be similarly computed, but we don't need these coefficients. Recall the $T[i, j]$ notation from Definition 4.1.

Proposition 6.4. *With the above notation, for location $\lambda = (r, s, t)$ in \triangle_d we have $c_t(\lambda) = 1$. Moreover, if $s > 0$,*

$$(6.3) \quad u = -(T[t+1, d-r])^{-1}v,$$

where $u = (c_{t+1}(\lambda), c_{t+2}(\lambda) \dots, c_{d-r}(\lambda))^t$ and $v = (T_{0t}, T_{1t} \dots, T_{s-1,t})^t$.

Proof. By Lemma 5.47 and Lemma 6.3, $c_t(\lambda) = 1$.

For the rest of the proof, assume that $s > 0$. By (6.1), $b_i(\lambda) = 0$ for $0 \leq i \leq s-1$. By (6.2), $c_i(\lambda) = 0$ for $0 \leq i \leq t-1$ and $d-r+1 \leq i \leq d$. Evaluating (4.4) using the above comments, we obtain

$$(6.4) \quad v + T[t+1, d-r]u = 0.$$

The matrix T is very good by Definition 4.14, so $T[t+1, d-r]$ is invertible. Solving (6.4) for u , we obtain (6.3). \square

Recall the B -value concept from Definition 5.31. Our next goal is to compute these values for the Billiard Array B .

Definition 6.5. Pick $\lambda = (r, s, t) \in \mathbb{Z}^3$. If $\lambda \in \triangle_d$, then define $T[\lambda]$ to be the submatrix $T[t, d-r]$ from Definition 4.1. If $\lambda \notin \triangle_d$, then define $T[\lambda]$ to be the empty set \emptyset .

For the rest of this section assume $d \geq 2$. For a location $\tau = (r, s, t) \in \triangle_{d-2}$, consider the corresponding white 3-clique in \triangle_d from Lemma 5.12. This white 3-clique consists of the locations

$$\lambda = (r, s+1, t+1), \quad \mu = (r+1, s, t+1), \quad \nu = (r+1, s+1, t).$$

Next, consider the vectors $\mu \pm \alpha, \mu \pm \beta, \mu \pm \gamma$, where α, β, γ are from (5.1). Note that $\lambda = \mu - \alpha$ and $\nu = \mu + \beta$. Moreover,

$$\begin{aligned} \mu + \alpha &= (r+2, s-1, t+1), & \mu - \alpha &= (r, s+1, t+1), \\ \mu + \beta &= (r+1, s+1, t), & \mu - \beta &= (r+1, s-1, t+2), \\ \mu + \gamma &= (r, s, t+2), & \mu - \gamma &= (r+2, s, t). \end{aligned}$$

The above vectors form a hexagon as follows:

$$\begin{array}{ccccc} & \mu + \beta & & \mu - \alpha & \\ \mu - \gamma & & \mu & & \mu + \gamma \\ & \mu + \alpha & & \mu - \beta & \end{array}$$

Theorem 6.6. *With the above notation, the B -value of the white 3-clique in \triangle_d that corresponds to τ is*

$$(6.5) \quad \frac{\det(T[\mu + \alpha]) \det(T[\mu + \beta]) \det(T[\mu + \gamma])}{\det(T[\mu - \alpha]) \det(T[\mu - \beta]) \det(T[\mu - \gamma])},$$

where we interpret $\det(\emptyset) = 1$.

Proof. Consider the following vectors:

$$\begin{array}{ccccc} & \mu - \alpha + \beta & & & \\ \mu + \beta & & \mu - \alpha & & \\ \mu - \gamma & & \mu & & \mu + \gamma \end{array}$$

We have three black 3-cliques with the following locations:

- (i) $\mu, \mu - \alpha, \mu + \gamma;$
- (ii) $\mu - \gamma, \mu + \beta, \mu;$
- (iii) $\mu + \beta, \mu - \alpha + \beta, \mu - \alpha.$

For notational convenience, let $\bar{\tau}$ denote the B -value of the white 3-clique in \triangle_d that corresponds to τ . By Lemma 5.40,

$$(6.6) \quad \bar{\tau} = \tilde{\mathcal{B}}_{\mu-\alpha, \mu} \tilde{\mathcal{B}}_{\mu, \mu+\beta} \tilde{\mathcal{B}}_{\mu+\beta, \mu-\alpha}.$$

We now show that

$$(6.7) \quad \tilde{\mathcal{B}}_{\mu-\alpha, \mu} = -1.$$

We apply Lemma 5.39 to the black 3-clique (i) and obtain

$$(6.8) \quad \mathcal{B}_{\mu-\alpha} + \tilde{\mathcal{B}}_{\mu-\alpha, \mu} \mathcal{B}_{\mu} + \tilde{\mathcal{B}}_{\mu-\alpha, \mu+\gamma} \mathcal{B}_{\mu+\gamma} = 0.$$

For convenience we rewrite (6.8) as

$$(6.9) \quad \mathcal{B}_{\mu-\alpha} - \mathcal{B}_{\mu} + (\tilde{\mathcal{B}}_{\mu-\alpha, \mu} + 1) \mathcal{B}_{\mu} + \tilde{\mathcal{B}}_{\mu-\alpha, \mu+\gamma} \mathcal{B}_{\mu+\gamma} = 0.$$

By Proposition 5.41,

$$(6.10) \quad \mathcal{B}_{\mu} \in \mathbb{F}v_{t+1} + \mathbb{F}v_{t+2} + \cdots + \mathbb{F}v_{d-r-1},$$

$$(6.11) \quad \mathcal{B}_{\mu-\alpha} \in \mathbb{F}v_{t+1} + \mathbb{F}v_{t+2} + \cdots + \mathbb{F}v_{d-r},$$

$$(6.12) \quad \mathcal{B}_{\mu+\gamma} \in \mathbb{F}v_{t+2} + \mathbb{F}v_{t+3} + \cdots + \mathbb{F}v_{d-r}.$$

By Proposition 6.4,

$$(6.13) \quad c_{t+1}(\mu) = 1, \quad c_{t+1}(\mu - \alpha) = 1.$$

By (6.10), (6.11) and (6.13),

$$(6.14) \quad \mathcal{B}_{\mu-\alpha} - \mathcal{B}_{\mu} \in \mathbb{F}v_{t+2} + \mathbb{F}v_{t+3} + \cdots + \mathbb{F}v_{d-r}.$$

By (6.9), (6.12) and (6.14),

$$(6.15) \quad (\tilde{\mathcal{B}}_{\mu-\alpha, \mu} + 1) \mathcal{B}_{\mu} \in \mathbb{F}v_{t+2} + \mathbb{F}v_{t+3} + \cdots + \mathbb{F}v_{d-r}.$$

By (6.10), the equation on the left in (6.13), and (6.15), we obtain (6.7).

Next, we show that

$$(6.16) \quad \tilde{\mathcal{B}}_{\mu, \mu+\beta} = \frac{\det(T[\mu + \alpha]) \det(T[\mu])}{\det(T[\mu - \beta]) \det(T[\mu - \gamma])}.$$

There are two cases. First assume that $s \neq 0$. We apply Lemma 5.39 to the black 3-clique (ii) and obtain

$$(6.17) \quad \mathcal{B}_{\mu} + \tilde{\mathcal{B}}_{\mu, \mu+\beta} \mathcal{B}_{\mu+\beta} + \tilde{\mathcal{B}}_{\mu, \mu-\gamma} \mathcal{B}_{\mu-\gamma} = 0.$$

By Proposition 5.41,

$$(6.18) \quad \mathcal{B}_{\mu-\gamma} \in \mathbb{F}v_t + \mathbb{F}v_{t+1} + \cdots + \mathbb{F}v_{d-r-2}.$$

By (6.17) and (6.18),

$$(6.19) \quad \mathcal{B}_{\mu} + \tilde{\mathcal{B}}_{\mu, \mu+\beta} \mathcal{B}_{\mu+\beta} \in \mathbb{F}v_t + \mathbb{F}v_{t+1} + \cdots + \mathbb{F}v_{d-r-2}.$$

By Proposition 5.41,

$$(6.20) \quad \begin{aligned} \mathcal{B}_{\mu} &\in \mathbb{F}v_{t+1} + \mathbb{F}v_{t+2} + \cdots + \mathbb{F}v_{d-r-1}; \\ \mathcal{B}_{\mu+\beta} &\in \mathbb{F}v_t + \mathbb{F}v_{t+1} + \cdots + \mathbb{F}v_{d-r-1}. \end{aligned}$$

By Proposition 6.4 and Cramer's rule,

$$(6.21) \quad \begin{aligned} c_{d-r-1}(\mu) &= \frac{(-1)^s \det(T[t+1, d-r-2])}{\det(T[t+2, d-r-1])}, \\ c_{d-r-1}(\mu + \beta) &= \frac{(-1)^{s+1} \det(T[t, d-r-2])}{\det(T[t+1, d-r-1])}. \end{aligned}$$

By (6.19)–(6.21),

$$(6.22) \quad \tilde{\mathcal{B}}_{\mu, \mu+\beta} = \frac{\det(T[t+1, d-r-2]) \det(T[t+1, d-r-1])}{\det(T[t+2, d-r-1]) \det(T[t, d-r-2])}.$$

Evaluating (6.22) using Definition 6.5, we obtain (6.16).

Next assume that $s = 0$. Using an argument similar to (6.17)–(6.21), we obtain

$$(6.23) \quad \tilde{\mathcal{B}}_{\mu, \mu+\beta} = \frac{\det(T[t+1, t+1])}{\det(T[t, t])}.$$

Since we interpret $\det(\emptyset) = 1$,

$$(6.24) \quad \det(T[\mu + \alpha]) = 1, \quad \det(T[\mu - \beta]) = 1.$$

Evaluate (6.23) using Definition 6.5. Combining the result with (6.24), we obtain (6.16).

We have shown (6.16). In a similar manner using the black 3-clique (iii), we obtain

$$(6.25) \quad \tilde{\mathcal{B}}_{\mu+\beta, \mu-\alpha} = -\frac{\det(T[\mu + \beta]) \det(T[\mu + \gamma])}{\det(T[\mu - \alpha]) \det(T[\mu])}.$$

Evaluating (6.6) using (6.7), (6.16), (6.25) we obtain (6.5). \square

7. HOW $\mathcal{T}_d(\mathbb{F})$ IS RELATED TO $BA_d(\mathbb{F})$

Recall the set $\mathcal{T}_d(\mathbb{F})$ from Definition 4.14, and the set $BA_d(\mathbb{F})$ from Definition 5.34. In this section we explain how $\mathcal{T}_d(\mathbb{F})$ is related to $BA_d(\mathbb{F})$. For convenience, we first consider an equivalence relation on $\mathcal{T}_d(\mathbb{F})$.

Definition 7.1. For $T, T' \in \mathcal{T}_d(\mathbb{F})$, we declare $T \sim T'$ whenever there exist invertible diagonal matrices $H, K \in \text{Mat}_{d+1}(\mathbb{F})$ such that

$$(7.1) \quad T' = HTK.$$

The relation \sim is an equivalence relation. For $T \in \mathcal{T}_d(\mathbb{F})$, let $[T]$ denote the equivalence class of \sim that contains T . Let $\mathbb{T}_d(\mathbb{F})$ denote the set of equivalence classes for \sim .

Recall the map $b : \mathcal{T}_d(\mathbb{F}) \rightarrow BA_d(\mathbb{F})$ from Definition 6.1. As we will see, b is surjective but not bijective. We will show that for $T, T' \in \mathcal{T}_d(\mathbb{F})$, $b(T) = b(T')$ if and only if $T \sim T'$. This tells us that the b -induced map $\mathbb{T}_d(\mathbb{F}) \rightarrow BA_d(\mathbb{F})$ is bijective.

Referring to Definition 7.1, assume that $T \sim T'$. Pick invertible diagonal matrices $H, K \in \text{Mat}_{d+1}(\mathbb{F})$ that satisfy (7.1). Observe that the entries $H_{ii} \neq 0$ and $K_{ii} \neq 0$ for $0 \leq i \leq d$. View T as the transition matrix from a basis $\{u_i\}_{i=0}^d$ of V to a basis $\{v_i\}_{i=0}^d$ of V as around (4.1). Recall the flags $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$ from Definition 6.1. The matrix T' can be viewed as the transition matrix from a basis $\{u'_i\}_{i=0}^d$ of V to a basis $\{v'_i\}_{i=0}^d$ of V , where $u_i = H_{ii}u'_i$ and $K_{ii}v_i = v'_i$ for $0 \leq i \leq d$. Observe that the flags $\{U_i\}_{i=0}^d$, $\{U'_i\}_{i=0}^d$, $\{U''_i\}_{i=0}^d$ are the same for T and T' .

Lemma 7.2. *Let matrices $T, T' \in \mathcal{T}_d(\mathbb{F})$ satisfy $T \sim T'$ in the sense of Definition 7.1. Then $b(T) = b(T')$.*

Proof. By the discussion above the lemma statement, along with Definition 6.1. \square

Definition 7.3. Using the map $b : \mathcal{T}_d(\mathbb{F}) \rightarrow BA_d(\mathbb{F})$, we define a map $\mathbf{b} : \mathbb{T}_d(\mathbb{F}) \rightarrow BA_d(\mathbb{F})$ as follows. Given an equivalence class $[T] \in \mathbb{T}_d(\mathbb{F})$, the image of $[T]$ under \mathbf{b} is $b(T)$. By Lemma 7.2 the map \mathbf{b} is well-defined.

Theorem 7.4. *The map $\mathbf{b} : \mathbb{T}_d(\mathbb{F}) \rightarrow BA_d(\mathbb{F})$ from Definition 7.3 is bijective.*

Proof. First we show that the map \mathbf{b} is surjective. Pick a Billiard Array B on V . It suffices to show that there exists $T \in \mathcal{T}_d(\mathbb{F})$ such that $B \in b(T)$. For $0 \leq i \leq d$, pick $0 \neq u_i \in B_\mu$ where $\mu = (d-i, i, 0) \in \Delta_d$, and $0 \neq v_i \in B_\nu$ where $\nu = (d-i, 0, i) \in \Delta_d$. By Definition 5.14, $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ are bases of V . Let $T \in \text{Mat}_{d+1}(\mathbb{F})$ denote the transition matrix from the basis $\{u_i\}_{i=0}^d$ to the basis $\{v_i\}_{i=0}^d$. By Corollary 5.44 $T \in \mathcal{T}_d(\mathbb{F})$. Above Lemma 6.2 we refer to a Billiard Array obtained from Theorem 5.22. By Lemma 6.2, this Billiard Array is the Billiard Array B . Hence $B \in b(T)$ by the last sentence above Lemma 6.2. We have shown that the map \mathbf{b} is surjective.

Next we show that the map \mathbf{b} is injective. Suppose that $T, T' \in \mathcal{T}_d(\mathbb{F})$ satisfy $b(T) = b(T')$. We will show that $T \sim T'$ in the sense of Definition 7.1. Define the Billiard Arrays $B \in b(T)$ and $B' \in b(T')$ as around Definition 6.1. Since $b(T) = b(T')$, the Billiard Arrays B and B' are isomorphic. Therefore there exists an isomorphism σ of Billiard Arrays from B to B' . By Definition 5.16, $\sigma : V \rightarrow V$ is an \mathbb{F} -linear bijection that sends $B_\lambda \mapsto B'_\lambda$ for all $\lambda \in \Delta_d$. Associated with T we have the bases $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ of V from Definition 6.1. Similarly, associated with T' we have bases $\{u'_i\}_{i=0}^d$ and $\{v'_i\}_{i=0}^d$ of V . To be more precise, T is the transition matrix from the basis $\{u_i\}_{i=0}^d$ to the basis $\{v_i\}_{i=0}^d$, and T' is the transition matrix from the basis $\{u'_i\}_{i=0}^d$ to the basis $\{v'_i\}_{i=0}^d$. Since σ is an \mathbb{F} -linear bijection, T is also the transition matrix from the basis $\{\sigma(u_i)\}_{i=0}^d$ of V to the basis $\{\sigma(v_i)\}_{i=0}^d$ of V . By Lemma 6.2, for $0 \leq i \leq d$ there exist nonzero $h_i, k_i \in \mathbb{F}$ such that $\sigma(u_i) = h_i u'_i$ and $k_i \sigma(v_i) = v'_i$. Define diagonal matrices $H, K \in \text{Mat}_{d+1}(\mathbb{F})$ such that $H_{ii} = h_i$ and $K_{ii} = k_i$ for $0 \leq i \leq d$. By construction H and K are invertible. By the above comments, $T' = HTK$. By Definition 7.1, $T \sim T'$. We have shown that the map \mathbf{b} is injective. Hence the map \mathbf{b} is bijective. \square

We continue to discuss the equivalence relation \sim from Definition 7.1. Our next goal is to identify a representative in each equivalence class.

Definition 7.5. A matrix $T \in \mathcal{T}_d(\mathbb{F})$ is called *nice* whenever $T_{0i} = T_{ii} = 1$ for $0 \leq i \leq d$.

Lemma 7.6. *For the equivalence relation \sim on $\mathcal{T}_d(\mathbb{F})$ from Definition 7.1, each equivalence class contains a unique nice element in the sense of Definition 7.5.*

Proof. First, we show that each equivalence class in $\mathcal{T}_d(\mathbb{F})$ contains at least one nice element. Choose $T \in \mathcal{T}_d(\mathbb{F})$. By Definition 4.14, we have $T_{0i} \neq 0$ and $T_{ii} \neq 0$ for $0 \leq i \leq d$. Define diagonal matrices $H, K \in \text{Mat}_{d+1}(\mathbb{F})$ such that $H_{ii} = T_{0i}/T_{ii}$ and $K_{ii} = 1/T_{0i}$ for $0 \leq i \leq d$. By construction H and K are invertible. Let $T' = HTK$. By construction, $T' \in \mathcal{T}_d(\mathbb{F})$ and $T'_{0i} = T'_{ii} = 1$ for $0 \leq i \leq d$. By Definition 7.1, we have $T \sim T'$. By Definition 7.5, T' is nice. Therefore each equivalence class in $\mathcal{T}_d(\mathbb{F})$ contains at least one nice element.

Next, we show that this nice element is unique. Suppose that T, T' are nice elements in $\mathcal{T}_d(\mathbb{F})$ and $T \sim T'$. By Definition 7.1, there exist invertible diagonal matrices $H, K \in \text{Mat}_{d+1}(\mathbb{F})$ that satisfy (7.1). For $0 \leq i \leq d$, examining the $(0, i)$ -entry in (7.1), we obtain $T'_{0i} = H_{00}T_{0i}K_{ii}$. Examining the (i, i) -entry in (7.1), we obtain $T'_{ii} = H_{ii}T_{ii}K_{ii}$. By Definition 7.5, we have $T_{0i} = T_{ii} = T'_{0i} = T'_{ii} = 1$. By the above comments, we have $K_{ii} = 1/H_{00}$ and $H_{ii} = H_{00}$ for $0 \leq i \leq d$. Therefore $H = H_{00}I$ and $K = I/H_{00}$. Consequently $T = T'$ by (7.1). We have shown that each equivalence class in $\mathcal{T}_d(\mathbb{F})$ contains a unique nice element. \square

8. A COMMUTATIVE DIAGRAM

Recall the set $\mathcal{T}_d(\mathbb{F})$ from Definition 4.14, the set $BA_d(\mathbb{F})$ from Definition 5.34, and the set $VF_d(\mathbb{F})$ from Definition 5.35. For the moment assume $d \geq 2$. In this section, we will describe how the sets $\mathcal{T}_d(\mathbb{F})$, $BA_d(\mathbb{F})$, $VF_d(\mathbb{F})$ and $VF_{d-2}(\mathbb{F})$ are related. In order to do this, we will establish a commutative diagram. As we proceed, some of our results do not require $d \geq 2$. So until further notice, assume $d \geq 0$. First we define a map $D : \mathcal{T}_d(\mathbb{F}) \rightarrow VF_d(\mathbb{F})$.

Definition 8.1. For $T \in \mathcal{T}_d(\mathbb{F})$, define a function $D(T) : \Delta_d \rightarrow \mathbb{F}$ as follows. For each location $\lambda = (r, s, t) \in \Delta_d$, the image of λ under $D(T)$ is $\det(T[\lambda])$, where $T[\lambda]$ is from Definition 6.5.

Lemma 8.2. *With reference to Definition 8.1, $D(T)$ is a value function on Δ_d in the sense of Definition 5.32. In other words, $D(T) \in VF_d(\mathbb{F})$.*

Proof. Pick $\lambda = (r, s, t) \in \Delta_d$. By construction $D(T)(\lambda) = \det(T[t, d-r])$, which is nonzero since T is very good. We have shown that $D(T)(\lambda) \neq 0$ for all $\lambda \in \Delta_d$. Therefore $D(T) \in VF_d(\mathbb{F})$. \square

Definition 8.3. We define a map $D : \mathcal{T}_d(\mathbb{F}) \rightarrow VF_d(\mathbb{F})$ as follows. For $T \in \mathcal{T}_d(\mathbb{F})$, the image of T under D is the function $D(T)$ from Definition 8.1. By Lemma 8.2 the map D is well defined.

For later use, we recall an elementary fact from linear algebra. Pick $T \in \text{Mat}_{d+1}(\mathbb{F})$. For $0 \leq i, j \leq d$, let $T^{(i,j)}$ denote the determinant of the $d \times d$ matrix that results from deleting the i -th row and the j -th column of T . Applying the Laplace expansion to the bottom row of T we obtain

$$(8.1) \quad \det(T) = \sum_{j=0}^d (-1)^{d+j} T_{dj} T^{(d,j)}.$$

Lemma 8.4. *The map $D : \mathcal{T}_d(\mathbb{F}) \rightarrow VF_d(\mathbb{F})$ from Definition 8.3 is bijective.*

Proof. For $f \in VF_d(\mathbb{F})$, we show that there exists a unique $T \in \mathcal{T}_d(\mathbb{F})$ such that $D(T) = f$. Our strategy is as follows. We are going to show that T exists. As we proceed, our calculation will show that there is only one solution for T . For $0 \leq i, j \leq d$, we solve for T_{ij} by induction on $i+j$. First assume that $i+j=0$, so that $i=j=0$. Then $D(T) = f$ forces $T_{00} = f((d, 0, 0))$. Next assume that $i+j > 0$. If $i > j$, then $T_{ij} = 0$ since T is required to be upper triangular. If $i = 0$, then $D(T) = f$ forces $T_{0j} = f((d-j, 0, j))$. If $0 < i \leq j$, then $D(T) = f$ forces $\det(T[j-i, j]) = f((d-j, i, j-i))$. Consider the matrix $T[j-i, j]$. The bottom right entry is T_{ij} . All the other entries have already been computed by induction. We now compute $\det(T[j-i, j])$ using the Laplace expansion to its

bottom row. Applying (8.1) to $T[j-i, j]$, we obtain a formula for $\det(T[j-i, j])$. In this formula the coefficient of T_{ij} is $\det(T[j-i, j-1])$, which is nonzero by assumption. Therefore there is a unique solution for T_{ij} . We have shown that the map D is bijective. \square

From now until the end of Theorem 8.6, assume that $d \geq 2$. Next we define a map $w : VF_d(\mathbb{F}) \rightarrow VF_{d-2}(\mathbb{F})$.

Definition 8.5. Assume $d \geq 2$. We define a map $w : VF_d(\mathbb{F}) \rightarrow VF_{d-2}(\mathbb{F})$ as follows. Given $f \in VF_d(\mathbb{F})$, we describe the image of f under w . For a location $\tau = (r, s, t) \in \Delta_{d-2}$, the value of $w(f)(\tau)$ is

$$(8.2) \quad \frac{f(\mu + \alpha)f(\mu + \beta)f(\mu + \gamma)}{f(\mu - \alpha)f(\mu - \beta)f(\mu - \gamma)}.$$

Here $\mu = (r+1, s, t+1) \in \Delta_d$ and α, β, γ are from (5.1). We interpret $f(\lambda) = 1$ for all $\lambda \in \mathbb{Z}^3$ such that $\lambda \notin \Delta_d$.

Theorem 8.6. Assume $d \geq 2$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T}_d(\mathbb{F}) & \xrightarrow{b} & BA_d(\mathbb{F}) \\ D \downarrow & & \downarrow \theta \\ VF_d(\mathbb{F}) & \xrightarrow{w} & VF_{d-2}(\mathbb{F}) \end{array}$$

Here θ is from Definition 5.36, b is from Definition 6.1, D is from Definition 8.3, and w is from Definition 8.5.

Proof. For $T \in \mathcal{T}_d(\mathbb{F})$ chase T around the diagram using Theorem 6.6. \square

Recall the equivalence relation \sim on $\mathcal{T}_d(\mathbb{F})$ from Definition 7.1.

Definition 8.7. Via the bijection $D : \mathcal{T}_d(\mathbb{F}) \rightarrow VF_d(\mathbb{F})$, the equivalence relation \sim on $\mathcal{T}_d(\mathbb{F})$ induces an equivalence relation on $VF_d(\mathbb{F})$, which we denote by \approx . In other words, for $T, T' \in \mathcal{T}_d(\mathbb{F})$, $T \sim T'$ if and only if $D(T) \approx D(T')$. For $f \in VF_d(\mathbb{F})$, let $[f]$ denote the equivalence class of \approx that contains f . Let $\mathbb{V}\mathbb{F}_d(\mathbb{F})$ denote the set of equivalence classes for \approx .

We now give an alternative description of \approx .

Lemma 8.8. For $f, f' \in VF_d(\mathbb{F})$ the following are equivalent:

- (i) $f \approx f'$ in the sense of Definition 8.7;
- (ii) there exist nonzero $h_i, k_i \in \mathbb{F}$ ($0 \leq i \leq d$) such that

$$(8.3) \quad f'(\lambda) = f(\lambda) \prod_{i=0}^s (h_i k_{t+i})$$

for $\lambda = (r, s, t) \in \Delta_d$.

Proof. By Lemma 8.4 there exist $T, T' \in \mathcal{T}_d(\mathbb{F})$ such that $D(T) = f$ and $D(T') = f'$.

(i) \Rightarrow (ii) By Definition 8.7 we have $T \sim T'$. So there exist invertible diagonal matrices $H, K \in \text{Mat}_{d+1}(\mathbb{F})$ that satisfy (7.1). Define $h_i = H_{ii}$ and $k_i = K_{ii}$ for

$0 \leq i \leq d$. By construction $0 \neq h_i, k_i \in \mathbb{F}$ for $0 \leq i \leq d$, and $T'_{ij} = h_i k_j T_{ij}$ for $0 \leq i, j \leq d$. Pick $\lambda = (r, s, t) \in \Delta_d$. By Definition 8.1 and the above comments,

$$\begin{aligned} f'(\lambda) &= \det(T'[t, d-r]) \\ &= h_0 h_1 \dots h_s k_t k_{t+1} \dots k_{s+t} \det(T[t, d-r]) \\ &= h_0 h_1 \dots h_s k_t k_{t+1} \dots k_{s+t} f(\lambda). \end{aligned}$$

Therefore (8.3) holds.

(ii) \Rightarrow (i) Define diagonal matrices $H, K \in \text{Mat}_{d+1}(\mathbb{F})$ such that $H_{ii} = h_i$ and $K_{ii} = k_i$ for $0 \leq i \leq d$. By construction H, K are invertible. We will show that $T' = HTK$. In order to do this, we show that $T'_{ij} = h_i k_j T_{ij}$ for $0 \leq i, j \leq d$. We proceed by induction on $i + j$. First assume that $i + j = 0$, so that $i = j = 0$. Applying (8.3) to $\lambda = (d, 0, 0) \in \Delta_d$ we obtain $T'_{00} = h_0 k_0 T_{00}$. Next assume that $i + j > 0$. If $i > j$, then $T'_{ij} = T_{ij} = 0$. Therefore $T'_{ij} = h_i k_j T_{ij}$. If $i = 0$, then by applying (8.3) to $\lambda = (d-j, 0, j) \in \Delta_d$ we obtain $T'_{0j} = h_0 k_j T_{0j}$. If $0 < i \leq j$, then by applying (8.3) to $\lambda = (d-j, i, j-i) \in \Delta_d$ we obtain

$$(8.4) \quad \det(T'[j-i, j]) = \det(T[j-i, j]) \prod_{l=0}^i (h_l k_{j-i+l}).$$

By (8.4) and induction we routinely obtain $T'_{ij} = h_i k_j T_{ij}$. We have shown that $T' = HTK$. By Definition 7.1, $T \sim T'$. By Definition 8.7, $f \approx f'$. \square

Definition 8.9. An element $f \in VF_d(\mathbb{F})$ is called *fine* whenever $f((i, d-i, 0)) = f((i, 0, d-i)) = 1$ for $0 \leq i \leq d$.

Lemma 8.10. For $T \in \mathcal{T}_d(\mathbb{F})$ the following are equivalent:

- (i) T is nice in the sense of Definition 7.5;
- (ii) $D(T)$ is fine in the sense of Definition 8.9.

Proof. By Definition 8.1,

$$(8.5) \quad D(T)((i, d-i, 0)) = T_{00} T_{11} \dots T_{d-i, d-i}. \quad (0 \leq i \leq d)$$

Similarly, by Definition 8.1,

$$(8.6) \quad D(T)((i, 0, d-i)) = T_{0, d-i}. \quad (0 \leq i \leq d)$$

First assume that T is nice. By Definition 7.5 along with (8.5) and (8.6), $D(T)((i, d-i, 0)) = D(T)((i, 0, d-i)) = 1$ for $0 \leq i \leq d$. Therefore $D(T)$ is fine in view of Definition 8.9.

Next assume that $D(T)$ is fine. By Definition 8.9 along with (8.5) and (8.6), $T_{0i} = T_{ii} = 1$ for $0 \leq i \leq d$. Therefore T is nice in view of Definition 7.5. \square

Corollary 8.11. Under the equivalence relation \approx from Definition 8.7, each equivalence class contains a unique fine element in $VF_d(\mathbb{F})$.

Proof. By Lemma 7.6 and Lemma 8.10. \square

Definition 8.12. We define a map $\mathbf{D} : \mathbb{T}_d(\mathbb{F}) \rightarrow \mathbb{VF}_d(\mathbb{F})$ as follows. Given an equivalence class $[T] \in \mathbb{T}_d(\mathbb{F})$, the image of $[T]$ under \mathbf{D} is $[D(T)]$. By Definition 8.7 the map \mathbf{D} is well-defined.

Lemma 8.13. The map $\mathbf{D} : \mathbb{T}_d(\mathbb{F}) \rightarrow \mathbb{VF}_d(\mathbb{F})$ from Definition 8.12 is bijective.

Proof. By Lemma 8.4 and Definition 8.7. \square

For the rest of this section assume $d \geq 2$.

Lemma 8.14. *Assume $d \geq 2$. Recall the map w from Definition 8.5. For $f, f' \in VF_d(\mathbb{F})$, suppose that $f \approx f'$ in the sense of Definition 8.7. Then $w(f) = w(f')$.*

Proof. Apply (8.2) to f' and evaluate the result using (8.3). \square

Definition 8.15. Assume $d \geq 2$. We define a map $\mathbf{w} : \mathbb{VF}_d(\mathbb{F}) \rightarrow VF_{d-2}(\mathbb{F})$ as follows. Given an equivalence class $[f] \in \mathbb{VF}_d(\mathbb{F})$, the image of $[f]$ under \mathbf{w} is $w(f)$. By Lemma 8.14 the map \mathbf{w} is well-defined.

Lemma 8.16. *Assume $d \geq 2$. Then the map $\mathbf{w} : \mathbb{VF}_d(\mathbb{F}) \rightarrow VF_{d-2}(\mathbb{F})$ from Definition 8.15 is bijective.*

Proof. We will show that for $g \in VF_{d-2}(\mathbb{F})$, there exists a unique fine $f \in VF_d(\mathbb{F})$ such that $w(f) = g$. Our strategy is as follows. We are going to show that f exists. As we proceed, our calculation will show that there is only one solution for f . For $\lambda = (r, s, t) \in \triangle_d$, we solve for $f(\lambda)$ by induction on $s - r$. First assume that $s - r = -d$, so that $\lambda = (d, 0, 0)$. By Definition 8.9, $f(\lambda) = 1$. Next assume that $s - r > -d$. If $s = 0$ or $t = 0$, then by Definition 8.9, $f(\lambda) = 1$. If $s \neq 0$ and $t \neq 0$, then by Definition 8.5,

$$(8.7) \quad g(\tau) = \frac{f(\mu + \alpha)f(\mu + \beta)f(\mu + \gamma)}{f(\mu - \alpha)f(\mu - \beta)f(\mu - \gamma)},$$

where $\tau = (r, s - 1, t - 1) \in \triangle_{d-2}$, $\mu = (r + 1, s - 1, t) \in \triangle_d$ and α, β, γ are from (5.1). In terms of r, s, t the equation (8.7) becomes

$$(8.8) \quad g(r, s - 1, t - 1) = \frac{f(r + 2, s - 2, t)f(r + 1, s, t - 1)f(r, s - 1, t + 1)}{f(r, s, t)f(r + 1, s - 2, t + 1)f(r + 2, s - 1, t - 1)}.$$

In the right hand side of (8.8), all the factors except for $f(\lambda) = f(r, s, t)$ have been determined by induction, and these factors are nonzero. Hence $f(\lambda) = f(r, s, t)$ is uniquely determined. We have shown that for $g \in VF_{d-2}(\mathbb{F})$, there exists a unique fine $f \in VF_d(\mathbb{F})$ such that $w(f) = g$. By this and Corollary 8.11, the map \mathbf{w} is bijective. \square

Theorem 8.17. *Assume $d \geq 2$. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{T}_d(\mathbb{F}) & \xrightarrow{\mathbf{b}} & BA_d(\mathbb{F}) \\ \mathbf{D} \downarrow & & \downarrow \theta \\ \mathbb{VF}_d(\mathbb{F}) & \xrightarrow{\mathbf{w}} & VF_{d-2}(\mathbb{F}) \end{array}$$

Here θ is from Definition 5.36, \mathbf{b} is from Definition 7.3, \mathbf{D} is from Definition 8.12, and \mathbf{w} is from Definition 8.15.

Proof. Use Theorem 8.6. \square

Remark 8.18. In the commutative diagram from Theorem 8.17, each map is bijective.

9. AN EXAMPLE

In this section, we will give an example to illustrate Theorem 6.6. We first recall some notation. Fix $0 \neq q \in \mathbb{F}$. For $n \in \mathbb{N}$ define

$$[n]_q = \sum_{i=0}^{n-1} q^i, \quad [n]_q! = \prod_{i=1}^n [i]_q.$$

We interpret $[0]_q = 0$ and $[0]_q! = 1$.

For $n, k \in \mathbb{Z}$ we define $\begin{bmatrix} n \\ k \end{bmatrix}_q$ as follows. For $0 \leq k \leq n$,

$$(9.1) \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

If $k < 0$ or $k > n$, then for notational convenience define $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$.

Remark 9.1. In the right-hand side of (9.1), for certain values of q the denominator may be equal to 0. However, by [2, Theorem 6.1] $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q with integral coefficients. So (9.1) is well defined no matter which q we have chosen.

We will use the following result.

Lemma 9.2. [2, Proposition 6.1] For $n, k \in \mathbb{N}$,

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

For the rest of this section assume $d \geq 2$. Recall the set $\mathcal{T}_d(\mathbb{F})$ from Definition 4.14 and the map b from Definition 6.1. We will display a matrix $T \in \mathcal{T}_d(\mathbb{F})$ which is nice in the sense of Definition 7.5. Then we show that for a Billiard Array $B \in b(T)$ the B -value of each white 3-clique in \triangle_d is equal to q^{-1} .

Define an upper triangular matrix $T \in \text{Mat}_{d+1}(\mathbb{F})$ as follows. For $0 \leq i \leq j \leq d$,

$$(9.2) \quad T_{ij} = \begin{bmatrix} j \\ i \end{bmatrix}_q.$$

Lemma 9.3. For $0 \leq i \leq d$, $T_{0i} = T_{ii} = 1$.

Proof. By (9.2). □

Lemma 9.4. For $1 \leq i \leq d$ and $0 \leq j \leq d-1$,

$$(9.3) \quad T_{i,j+1} - T_{ij} = q^{j-i+1} T_{i-1,j}.$$

Proof. By (9.2) and Lemma 9.2. □

For later use, we recall an elementary fact from linear algebra. Pick $A \in \text{Mat}_{d+1}(\mathbb{F})$. For $0 \leq i \leq d$, let A_i denote the i -th column of A . Let A' denote the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with i -th column $A_i - A_{i-1}$ for $1 \leq i \leq d$ and 0-th column A_0 . Then

$$(9.4) \quad \det(A) = \det(A').$$

For $0 \leq i \leq j \leq d$, recall the submatrix $T[i, j]$ from Definition 4.1. We now compute $\det(T[i, j])$.

Lemma 9.5. For $0 \leq i < j \leq d$,

$$\det(T[i, j]) = q^{i(j-i)} \det(T[i, j-1]).$$

Proof. Apply (9.4) to $A = T[i, j]$ and use (9.3). \square

Proposition 9.6. For $0 \leq i \leq j \leq d$,

$$\det(T[i, j]) = q^{i(j-i)(j-i+1)/2}.$$

Proof. Use Lemma 9.5 and induction on $j - i$. \square

We now restate Proposition 9.6 using the notation $T[\lambda]$ from Definition 6.5.

Proposition 9.7. For $\lambda = (r, s, t) \in \Delta_d$,

$$(9.5) \quad \det(T[\lambda]) = q^{ts(s+1)/2},$$

where $T[\lambda]$ is from Definition 6.5.

Proof. Use Definition 6.5 and Proposition 9.6. \square

Lemma 9.8. T is very good in the sense of Definition 4.4. Moreover, $T \in \mathcal{T}_d(\mathbb{F})$.

Proof. By construction T is upper triangular. By Proposition 9.6, $\det(T[i, j]) \neq 0$ for $0 \leq i \leq j \leq d$. Therefore T is very good by Definition 4.4. Consequently $T \in \mathcal{T}_d(\mathbb{F})$. \square

Lemma 9.9. T is nice in the sense of Definition 7.5.

Proof. By Lemmas 9.3, 9.8. \square

Since $T \in \mathcal{T}_d(\mathbb{F})$, we can apply the map b to T . Recall from Definition 6.1 that $b(T)$ is an isomorphism class of Billiard Arrays. For a Billiard Array $B \in b(T)$, we compute the B -value of each white 3-clique in Δ_d .

Proposition 9.10. With the above notation, the B -value of each white 3-clique in Δ_d is equal to q^{-1} .

Proof. Apply Theorem 6.6 to the Billiard Array B and evaluate (6.5) using (9.5). \square

Combining Proposition 9.10 with Theorem 7.4 and Lemma 7.6, we obtain the following result.

Corollary 9.11. T is the unique nice matrix in $\mathcal{T}_d(\mathbb{F})$ such that for a Billiard Array $B \in b(T)$ the B -value of each white 3-clique in Δ_d is equal to q^{-1} .

Remark 9.12. Assume $q = 1$. Then the expression $\begin{bmatrix} n \\ k \end{bmatrix}_q$ from (9.1) becomes the usual binomial coefficient $\binom{n}{k}$. So for $0 \leq i \leq j \leq d$, $T_{ij} = \binom{j}{i}$ by (9.2) and $\det(T[i, j]) = 1$ by Proposition 9.6. By Proposition 9.10, for a Billiard Array $B \in b(T)$ the B -value of each white 3-clique in Δ_d is equal to 1.

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